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Stability of a Random Diffusion with Linear Drift

Gopal K. Basak* and Arnab Bisi†

Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

Mrinal K. Ghosh‡

Department of Mathematics, Indian Institute of Science, Bangalore 560 012, India

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We consider a linear system with Markovian switching which is perturbed by Gaussian type noise. If the linear system is mean square stable then we show that under certain conditions the perturbed system is also stable. We also show that under certain conditions the linear system with Markovian switching can be stabilized by such noisy perturbation.

1. INTRODUCTION

Consider the linear stochastic system

$$dX(t) = A(\theta(t))X(t)\,dt, \qquad X(0) = X_0 \in \mathbb{R}^d, \tag{1.1}$$

where $\theta(t)$ is a continuous time Markov chain taking values in a finite set $\{1, 2, \ldots, N\}$, $A(i)$, $i = 1, 2, \ldots, N$, are $d \times d$ matrices. The system (1.1) is a typical example of a piecewise deterministic system which arises quite often in practice in systems with multiple modes, e.g., fault tolerant control systems, multiple target tracking, flexible manufacturing systems, etc. [7].

* Research partially supported by Grant D4276 DAG 94/95.SC27. E-mail address: mabasak@uxmail.ust.hk.

† E-mail address: mabisi@uxmail.ust.hk.

‡ E-mail address: mkg@math.iisc.ernet.in.

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Due to the presence of the Markovian switching parameter $\theta(t)$, the stability of the system (1.1) is quite involved [2, 6, 7]. Even if all the matrices $A(i)$, $i = 1, 2, \dots, N$, are stable the system (1.1) may not be stable. On the other hand the system (1.1) may be stable even if all the matrices $A(i)$, $i = 1, 2, \dots N$, are unstable [2]. Quite often due to various uncertainties in the environment and within the system, (1.1) gets perturbed and is modeled as

$$dX(t) = A(\theta(t))X(t)\,dt + \sigma(X(t), \theta(t))\,dW(t)$$
$$X(0) = X_0 \in \mathbb{R}^d, \tag{1.2}$$

where $\sigma(\cdot, \cdot)$ is a $d \times d$ matrix and $W(\cdot)$ is a standard d-dimensional Brownian motion.

The following question now arises. If the system (1.1) is stable (in an appropriate sense) would the system (1.2) remain stable? If $\sigma(\cdot, \cdot)$ is independent of (x, i) and is positive definite then the answer is always in the affirmative. But if $\sigma(\cdot, \cdot)$ does depend on (x, i) then the answer would depend on certain properties of $\sigma(\cdot, \cdot)$. In this paper we investigate this problem for both the nondegenerate and degenerate cases. We also address the following problem. Suppose the system (1.1) is unstable; can we add a suitable "noise" to it to make it stable? In other words, can (1.2) be stable even if (1.1) is not so? Under certain conditions on $\sigma(\cdot, \cdot)$ we answer this question in the affirmative.

2. PROBLEM DESCRIPTION AND PRELIMINARIES

Let $\theta(t)$ be a continuous-time Markov chain taking values in a finite set $\Theta = \{1, 2, \dots, N\}$ with generator $[\lambda_{ij}]$, $\lambda_{ij} > 0$, $i \neq j$. Consider the following linear system with Markovian switching

$$dX(t) = A(\theta(t))X(t)\,dt, \qquad X(0) = X_0 \in \mathbb{R}^d, \tag{2.1}$$

where $A(i)$, $i = 1, 2, \dots, N$, are $d \times d$ matrices. We would often write A_i for $A(i)$.

DEFINITION 2.1. We say that the system (2.1) is mean square stable (MSS) if for all $x \in \mathbb{R}^d$, $i \in \Theta$, we have

$$E_{x,i}[X'(t)X(t)] \to 0 \qquad \text{as } t \to \infty, \tag{2.2}$$

where $E_{x,i}$ denotes the expectation after conditioning on $\{X(0) = x, \theta(0) = i\}$. It can be shown that for the system (2.1), MSS is equivalent to the following: there exists a constant C such that

$$\lim_{T\to\infty} E_{x,i}\left[\int_0^T X'(t)X(t)\,dt\right] \le C|x|^2. \tag{2.3}$$

We now present two characterizations of the MSS of (2.1). For proof and other details we refer to [7, Chap. 2].

PROPOSITION 2.1. *The system* (2.1) *is MSS if and only if for any given positive definite symmetric matrices* Q_i, $i = 1, 2, \dots, N$, *the following system of matrix equations*

$$A_i'P_i + P_iA_i + \sum_{j=1}^{N} \lambda_{ij}P_j = -Q_i, \qquad i = 1, \dots, N \tag{2.4}$$

admits a unique set of positive definite symmetric solution P_i, $i = 1, \dots, N$.

PROPOSITION 2.2. *The system* (2.1) *is MSS if and only if all the eigenvalues of the matrix*

$$\tilde{A} := \operatorname{diag}(A_i \otimes I_d + I_d \otimes A_i)_{i=1,\dots,N} + \Lambda \otimes I_{d^2} \tag{2.5}$$

have negative real parts, where $\otimes$ *stands for the Kronecker product of matrices and* I_r *represents the identity matrix of order* $r \times r$.

We now consider a perturbed version of the system (2.1). Let $X(\cdot)$ be a d-dimensional random diffusion given by

$$\begin{aligned} dX(t) &= A(\theta(t))X(t)\,dt + \sigma(X(t), \theta(t))\,dW(t) \\ X(0) &= X_0 \in \mathbb{R}^d, \end{aligned} \tag{2.6}$$

where $W(\cdot)$ is a standard d-dimensional Wiener process independent of $\theta(\cdot)$ and $\sigma(\cdot,\cdot)$: $\mathbb{R}^d \times \Theta \to S^d$ (the space of $d \times d$ matrices).

We make the following assumption on σ.

(A1) There exists a constant $k_0 > 0$ such that

$$\|\sigma(x,i) - \sigma(y,i)\| \le k_0|x - y| \tag{2.7}$$

for all $x, y \in \mathbb{R}^d$, $i \in \Theta$.

Let $Y(t)$ denote the $\mathbb{R}^d \times \Theta$-valued process $(X(t), \theta(t))$. Then $Y(t)$ is a (time homogeneous) Markov process. Let $p(t, x, i, dy \times \{j\})$ denote the transition probability of the process $Y(t)$.

To design a linear quadratic regulator the system in (2.1) is used and the mean square stability is more relevant there. Therefore, throughout the paper MSS is studied for the system (2.1). However, when the perturbed system (2.6) is considered it is more relevant to study stochastic stability as in Has'minskii [5] or, more generally, the notion of existence of a unique invariant measure. Notice that MSS implies that the unique invariant is the Dirac measure at zero. If one allows the added noise to be degenerate, existence of a unique invariant measure also is not enough to study the system. For the case $d = 1$, $b(x, i) = x$, and $\sigma(x, i) = x$, the unique invariant is the Dirac measure at zero but the system (2.6) blows off to infinity if it starts from any other point than zero. Therefore, to study the system in (2.6) where degeneracy of the noise is allowed, we introduce, following Basak and Bhattacharya [1], the notion of stability in distribution for the process $Y(t)$.

DEFINITION 2.2. The process $Y(t)$ is said to be stable in distribution if there exists a probability measure $\pi(\cdot \times \{\cdot\})$ such that its transition probability $p(t, x, i, dy \times \{j\})$ converges weakly to $\pi(dy \times \{j\})$ as $t \to \infty$ for every $(x, i) \in \mathbb{R}^d \times \Theta$.

It is clear that the stability of $Y(t)$ in distribution implies the existence of a unique invariant probability measure for $Y(t)$. We conclude this section by recalling that a continuous time Markov chain $\theta(\cdot)$ with generator $\Lambda = [\lambda_{ij}]$ can be represented as a stochastic integral with respect to a Poisson random measure. Indeed, let Δ_{ij} be consecutive (with respect to the lexicographic ordering on $\Theta \times \Theta$), left closed, right open intervals of the real line each having length λ_{ij}. Define a function

$$h\colon \Theta \times \mathbb{R} \to \mathbb{R}$$

by

$$h(i, y) = \begin{cases} j - i, & \text{if } y \in \Delta_{ij} \\ 0, & \text{otherwise}. \end{cases} \tag{2.8}$$

Then

$$d\theta(t) = \int_{\mathbb{R}} h(\theta(t-), y)\, \nu(dt, dy) \tag{2.9}$$

$$\theta(0) = \theta_0,$$

where $\nu(dt, dy)$ is a Poisson random measure with intensity $dt \times m(dy)$, m being the Lebesgue measure on $\mathbb{R}$. For more information see [4, 8]. This representation will be useful in the next section.

3. STABILITY RESULTS

In this section we will derive the stability results for the process $Y(t) = (X(t), \theta(t))$. To this end we introduce the notion of asymptotic flatness of the flow $X(t)$. Let $X^{x,i}(t)$ denote the solution of (2.6) with initial condition $X(0) = x$, $\theta(0) = i$.

DEFINITION 3.1. We say that the flow $\{X^{x,i}(t), t \geq 0, x \in \mathbb{R}^d, i \in \Theta\}$ is asymptotically flat in the pth mean ($p > 0$) if for every compact $K \subset \mathbb{R}^d$,

$$\lim_{t\to\infty} \sup_{x,y\in K} E\left|X^{x,i}(t) - X^{y,i}(t)\right|^p = 0. \tag{3.1}$$

To achieve the asymptotic flatness we make the following assumption. We write

$$a(x,i) = \sigma(x,i)\sigma'(x,i)$$
$$a(x,y,i) = (\sigma(x,i) - \sigma(y,i))(\sigma(x,i) - \sigma(y,i))'.$$

(A2) Assume that there exist symmetric positive definite matrices B_i, $i = 1, \ldots, N$, and positive constants $\gamma > 0$, $k > 0$ such that for $x \neq y \in \mathbb{R}^d$, $i \in \Theta$,

$$\begin{aligned}
&2\big(B_i(x-y)\big)'\big(A_i(x-y)\big) - \frac{(2-k)\big(B_i(x-y)\big)'a(x,y,i)\big(B_i(x-y)\big)}{(x-y)'B_i(x-y)} \\
&\qquad + \operatorname{tr}\big(a(x,y,i)B_i\big) \\
&\qquad + \frac{2}{k}\big((x-y)'B_i(x-y)\big)^{1-k/2}\sum_{j=1}^{N}\lambda_{ij}\big((x-y)'B_j(x-y)\big)^{k/2} \\
&\quad \leq -\gamma|x-y|^2.
\end{aligned} \tag{3.2}$$

The examples discussed at the end of this section will show that the assumptions (A1) and (A2) arise quite naturally.

We shall now establish the asymptotic flatness of the system defined in (3.1).

LEMMA 3.1. *Under* (A1) *and* (A2) *the random diffusion* (2.6) *is asymptotically flat in the kth mean.*

Proof. Consider the Liapunov function

$$w(x,i) = (x'B_i x)^{k/2}, \qquad x \in \mathbb{R}^d, \quad i \in \Theta,$$

where $k > 0$, B_i are as in (3.2). For $0 < k < 2$, the above function can be suitably modified near the origin to make it a C^2-function in x for each

$i \in \Theta$. For $x \neq y \in \mathbb{R}^d$, $i \in \Theta$, set

$$\begin{aligned} Z^{x,y,i}(t) &= X^{x,i}(t) - X^{y,i}(t) \\ &= x - y + \int_0^t A\big(\theta^i(s)\big) Z^{x,y,i}(s)\, ds \\ &\quad + \int_0^t \Big(\sigma\big(X^{x,i}(s), \theta^i(s)\big) - \sigma\big(X^{y,i}(s), \theta^i(s)\big)\Big)\, dW(s), \end{aligned} \tag{3.3}$$

where the $\theta^i(\cdot)$ denote the Markov chain with $\theta(0) = i \in \Theta$.

Let N_0 be a positive integer such that $|x| < N_0/4$, $|y| < N_0/4$, and $|x-y| < N_0/4$. Let N be a positive integer such that $N \geq N_0$. Define the stopping times

$$\eta_0 = \inf\big\{t > 0 \colon X^{x,i}(t) = X^{y,i}(t)\big\},$$

$$\eta_N = \inf\big\{t > 0 \colon \big|X^{x,i}(t) - X^{y,i}(t)\big| = N\big\}.$$

Let $t_N = \eta_N \wedge \eta_0 \wedge t$. Applying Itô's formula to (3.3), we have for $N \geq N_0$

$$\begin{aligned} & w\big(Z^{x,y,i}(t_N), \theta^i(t_N)\big) - w(x-y, i) \\ &\quad = \int_0^{t_N} \tilde{L} w\big(Z^{x,y,i}(s), \theta^i(s)\big)\, ds \\ &\qquad + \int_0^{t_N} \Big(\nabla w\big(Z^{x,y,i}(s), \theta^i(s)\big)\Big)' \\ &\qquad\quad \times \Big(\sigma\big(X^{x,i}(s), \theta^i(s)\big) - \sigma\big(X^{y,i}(s), \theta^i(s)\big)\Big)\, dW(s) \\ &\qquad + \int_0^{t_N} \int_{\mathbb{R}} \Big(w\big(Z^{x,y,i}(s), i + h\big(\theta^i(s), l\big)\big) \\ &\qquad\qquad - w\big(Z^{x,y,i}(s), \theta^i(s)\big)\Big)\mu(ds, dl), \end{aligned} \tag{3.4}$$

where the function h is as in (2.8) and $\mu(ds, dl) = \nu(ds, dl) - m(dl)\, ds$ is the centered Poisson measure, $\tilde{L}$ is the differential generator of $(Z^{x,y,i}(\cdot), \theta^i(\cdot))$. Taking expectation in (3.4) we get

$$Ew\big(Z^{x,y,i}(t_N), \theta^i(t_N)\big) = w(x-y, i) + E\int_0^{t_N} \tilde{L} w\big(Z^{x,y,i}(s), \theta^i(s)\big)\, ds. \tag{3.5}$$

Now,

$$
\begin{aligned}
&\tilde{L}w(x-y,i)\\
&\quad = \frac{k}{2}\big((x-y)'B_i(x-y)\big)^{k/2-1}\\
&\qquad \times\Bigg[2\big(B_i(x-y)\big)'\big(A_i(x-y)\big) + \operatorname{tr}\big(a(x,y,i)B_i\big)\\
&\qquad\quad - \frac{(2-k)\big(B_i(x-y)\big)'a(x,y,i)B_i(x-y)}{(x-y)'B_i(x-y)}\\
&\qquad\quad + \frac{2}{k}\big((x-y)'B_i(x-y)\big)^{1-k/2}\\
&\qquad\qquad \times \sum_{j=1}^{N}\lambda_{ij}\big((x-y)'B_j(x-y)\big)^{k/2}\Bigg]. \qquad (3.6)
\end{aligned}
$$

Using (A2) in (3.6) it follows that for some $\delta > 0$

$$
\tilde{L}w(x-y,i) \le -\delta w(x-y,i). \qquad (3.7)
$$

From (3.7) it is easily seen that the limit as $N \to \infty$ in each term in (3.5) exists. We claim that for some $\mu_0 > 0$

$$
E\Big[w^{1+2\mu_0/k}\big(Z^{x,y,i}(t\wedge\eta_0),\theta^i(t\wedge\eta_0)\big)\Big] < \infty. \qquad (3.8)
$$

Indeed, we have for any $\mu_1 > 0$

$$
\begin{aligned}
&\tilde{L}w^{1+2\mu_1/k}(x-y,i)\\
&\quad = \left(\frac{k}{2}+\mu_1\right)\big((x-y)'B_i(x-y)\big)^{k/2+\mu_1-1}\\
&\qquad \times\Bigg[2\big(B_i(x-y)\big)'\big(A_i(x-y)\big) + \operatorname{tr}\big(a(x,y,i)B_i\big)\\
&\qquad\quad - \frac{(2-k-2\mu_1)\big(B_i(x-y)\big)'a(x,y,i)B_i(x-y)}{(x-y)'B_i(x-y)}\\
&\qquad\quad + \frac{2}{(k+2\mu_1)}\big((x-y)'B_i(x-y)\big)^{1-k/2-\mu_1}\\
&\qquad\quad \times \sum_{j=1}^{N}\lambda_{ij}\big((x-y)'B_j(x-y)\big)^{k/2+\mu_1}\Bigg]. \qquad (3.9)
\end{aligned}
$$

Now, for any $\mu_1 > 0$

$$k - 2\mu_1 \le \frac{k^2}{k + 2\mu_1} \le k.$$

Therefore,

$$\frac{\lambda_{ii} k}{k + 2\mu_1} \le \lambda_{ii}\left(1 - \frac{2\mu_1}{k}\right) \qquad \text{and} \qquad \frac{\lambda_{ij} k}{k + 2\mu_1} \le \lambda_{ij}, \quad \text{for } i \ne j.$$

Since the B_j's are all positive definite matrices, there exist positive constants C^B, $C_B > 0$, such that, for any l, $k = 1, \dots, n$ and $x \ne y$,

$$C_B \le \frac{(x-y)' B_l (x-y)}{(x-y)' B_k (x-y)} \le C^B.$$

Using these, we get

$$\begin{aligned}
&\frac{2\big((x-y)'B_i(x-y)\big)^{1-k/2-\mu_1}}{k + 2\mu_1} \sum_{j=1}^{N} \lambda_{ij}\big((x-y)'B_j(x-y)\big)^{k/2+\mu_1} \\
&\quad \le \tfrac{2}{k}(C^B)^{\mu_1}\big((x-y)'B_i(x-y)\big)^{1-k/2} \\
&\qquad \times \sum_{j=1_{j\ne i}}^{N} \lambda_{ij}\big((x-y)'B_j(x-y)\big)^{k/2} \\
&\qquad + \frac{2}{k}(C_B)^{\mu_1}\left(1 - \frac{2\mu_1}{k}\right)\lambda_{ii}(x-y)'B_i(x-y) \\
&\quad = \frac{2}{k}\big((x-y)'B_i(x-y)\big)^{1-k/2} \sum_{j=1}^{N} \lambda_{ij}\big((x-y)'B_j(x-y)\big)^{k/2} \\
&\qquad + \frac{2}{k}\big((C^B)^{\mu_1} - 1\big)\big((x-y)'B_i(x-y)\big)^{1-k/2} \\
&\qquad \times \sum_{j=1_{j\ne i}}^{N} \lambda_{ij}\big((x-y)'B_j(x-y)\big)^{k/2} \\
&\qquad + \frac{2}{k}\left((C_B)^{\mu_1}\left(1 - \frac{2\mu_1}{k}\right) - 1\right)\lambda_{ii}(x-y)'B_i(x-y).
\end{aligned} \tag{3.10}$$

Also,

$$\frac{2\mu_1(B_i(x-y))'a(x,y,i)B_i(x-y)}{(x-y)B_i(x-y)} \le 2\mu_1\eta_i|x-y|^2 \tag{3.11}$$

for some constant $\eta_i > 0$. Using (A2), (3.10), and (3.11) in (3.9), we get

$$\begin{aligned}
&\tilde{L}w^{1+2\mu_1/k}(x-y,i)\\
&\quad\le \left(\frac{k}{2}+\mu_1\right)\left((x-y)'B_i(x-y)\right)^{k/2+\mu_1-1}\\
&\qquad\times\Bigg(-\gamma+2\mu_1\eta_i+\frac{2}{k}\Bigg[\left|(C^B)^{\mu_1}-1\right|\\
&\qquad\qquad+\left|(C_B)^{\mu_1}\left(1-\frac{2\mu_1}{k}\right)-1\right|\Bigg]\zeta_B\sum_{j=1_{j\ne i}}^{N}\lambda_{ij}\Bigg)|x-y|^2,
\end{aligned}\tag{3.12}$$

where ζ_B is the largest of the eigenvalues of B_l's. Choosing $\mu_1 > 0$ sufficiently small in the above expression it follows that we can find a constant $\delta' > 0$ such that

$$\tilde{L}w^{1+2\mu_1/k}(x-y,i) \le -\delta' w^{1+2\mu_1/k}(x-y,i). \tag{3.13}$$

Then by Itô's formula

$$\begin{aligned}
&E\left[w^{1+2\mu_1/k}\left(Z^{x,y,i}(t_N),\theta^i(t_N)\right)\right]\\
&\qquad-E\left[\int_0^{t_N}\tilde{L}w^{1+2\mu_1/k}\left(Z^{x,y,i}(s),\theta^i(s)\right)ds\right]\\
&\quad= w^{1+2\mu_1/k}(x-y,i).
\end{aligned}\tag{3.14}$$

In view of (3.13), each term in (3.14) is nonnegative. Thus

$$\sup_{N\ge N_0} Ew^{1+2\mu_1/k}\left(Z^{x,y,i}(t_N),\theta^i(t_N)\right) < \infty.$$

Hence for any $0 < \mu_0 < \mu_1$, $w^{1+2\mu_0/k}(Z^{x,y,i}(t_N),\theta^i(t_N))$ is uniformly integrable for $N \ge N_0$. Therefore, letting $N \to \infty$ in (3.14), using monotone convergence theorem on the second term, we get

$$\begin{aligned}
&Ew^{1+2\mu_0/k}\left(Z^{x,y,i}(t\wedge\eta_0),\theta^i(t\wedge\eta_0)\right)\\
&\qquad-E\left[\int_0^{t\wedge\eta_0}\tilde{L}w^{1+2\mu_0/k}\left(Z^{x,y,i}(s),\theta^i(s)\right)ds\right]\\
&\quad= w^{1+2\mu_0/k}(x-y,i),
\end{aligned}\tag{3.15}$$

whence (3.8) follows. Now using (3.8) we let $N \to \infty$ in (3.5) to obtain

$$\begin{aligned} &Ew\big(Z^{x,y,i}(t \wedge \eta_0), \theta^i(t \wedge \eta_0)\big) \\ &\qquad = w(x-y,i) + E\left[\int_0^{t\wedge \eta_0} \tilde{L}w\big(Z^{x,y,i}(s), \theta^i(s)\big)\, ds\right]. \end{aligned} \tag{3.16}$$

Consider the process $Y(t) := \exp(\delta t) w(Z^{x,y,i}(t), \theta^i(t))$. It follows from (3.7) and (3.16) that $\{Y(t \wedge \eta_0),\ t \geq 0\}$ is a positive supermartingale with respect to the filtration

$$\mathscr{F}_t := \sigma\big\{W(s), \nu(A,B) \colon s \leq t,\ A \in \mathscr{B}([0,t]),\ B \in \mathscr{B}(\mathbb{R})\big\}. \tag{3.17}$$

Therefore

$$EY(t \wedge \eta_0) \leq EY(0) = w(x-y,i).$$

Since $Z^{x,y,i}(t) = 0$ a.e. for $t \geq \eta_0$, it follows that for all $t \geq 0$.

$$EY(t) \leq w(x-y,i). \tag{3.18}$$

Finally, using the positive definiteness of B_i's it follows from (3.18) that

$$E\big|X^{x,i}(t) - X^{y,i}(t)\big|^k \leq C_1 e^{-C_2 t}|x-y|^k, \tag{3.19}$$

where $C_1, C_2 > 0$ are some constants. The desired result follows from (3.19). ∎

COROLLARY 3.1. *Suppose that the system* (2.1) *is MSS. Assume* (A1) *and suppose that* $dk_0^2 < 1/\Lambda_P$, *where* k_0 *is as in* (2.7) *and* $\Lambda_P = \max(\Lambda_{P_1}, \dots, \Lambda_{P_N})$, Λ_{P_i} *being the largest eigenvalue of the matrix* P_i, *as in Proposition* 2.1, *with*

$$Q_i = I, \text{ for } i = 1, \dots, N. \tag{3.20}$$

Then the random diffusion (2.6) *is asymptotically flat in the mean square*.

Proof. Since the system (2.1) is MSS, we recall from Proposition 2.1 that the P_i's are symmetric positive definite and satisfy

$$A_i' P_i + P_i A_i + \sum_{j=1}^N \lambda_{ij} P_j = -I, \qquad 1 \leq i \leq N. \tag{3.21}$$

Using (3.21), we get

$$
\begin{aligned}
&2P_i x.A_i x + x.\sum_{j=1}^{N} \lambda_{ij} P_j x \\
&\qquad = x.\big(P_i A_i + A_i^T P_i\big)x + x.\sum_{j=1}^{N} \lambda_{ij} P_j x \\
&\qquad = -|x|^2.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\operatorname{tr}\big(a(x,y,i)P_i\big) &= \operatorname{tr}\Big(\sqrt{P_i}\,a(x,y,i)\sqrt{P_i}\Big) \\
&\le d\Big\|\sqrt{P_i}\,a(x,y,i)\sqrt{P_i}\Big\| \\
&\le d\|P_i\|\,\|a(x,y,i)\| \\
&\le d\Lambda_P k_0^2|x-y|^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&2\big(P_i(x-y)\big)'A_i(x-y) + (x-y)'\sum_{j=1}^{N} \lambda_{ij} P_j(x-y) + \operatorname{tr}\big(a(x,y,i)P_i\big) \\
&\qquad \le -|x-y|^2 + d\Lambda_P k_0^2|x-y|^2 \\
&\qquad = -\big(1 - d\Lambda_P k_0^2\big)|x-y|^2 \\
&\qquad = -\gamma'|x-y|^2, \qquad (3.22)
\end{aligned}
$$

where $\gamma' = (1 - d\Lambda_P k_0^2) > 0$ by our assumption. From (3.22) it follows that (A2) is satisfied with $B_i = P_i$, $i = 1, \ldots, N$, and $k = 2$. Hence the desired result. ∎

Under the following condition we will derive a growth property of the flow $X^{x,i}(t)$.

(A2)′ There exist symmetric positive definite matrices D_i, $i = 1, \ldots, N$, and constants $\beta > 0$, $\beta' > 0$, $k_0 > 0$ such that for $x \in \mathbb{R}^d$, $i \in \Theta$,

$$
\begin{aligned}
&2(D_i x)' \cdot (A_i x) + \operatorname{tr}\big(a(x,i)D_i\big) - \frac{(2-k_0)(D_i x)'a(x,i)\cdot D_i x}{x'D_i x} \\
&\qquad + \frac{2}{k_0}(x'D_i x)^{1-k_0/2}\sum_{j=1}^{N} \lambda_{ij}\big(x'D_j x\big)^{k_0/2} \\
&\quad \le -\beta|x|^2 + \beta'. \qquad (3.23)
\end{aligned}
$$

Remark 3.1. By taking $y = 0$ in (3.2), it is easy to see that (3.23) is satisfied with $B_i = D_i$ for some $\beta > 0$, $\beta' > 0$, and $k = k_0$. Thus (A2) $\Rightarrow$ (A2)$'$.

LEMMA 3.2. *Under* (A1), (A2)$'$, *there exists a* $\delta > 0$ *such that for* $x \in \mathbb{R}^d$, $i \in \Theta$

$$E\left|X^{x,i}(t)\right|^{k_0+\delta} < \infty \tag{3.24}$$

uniformly in $t \geq 0$.

Proof. Consider the Liapunov function

$$w_1(x,i) = (x' D_i x)^{k_0/2+\delta_0}$$

for some $\delta_0 > 0$ to be chosen later. For each $i \in \Theta$, $w_1(\cdot, i)$ may be modified near the origin to make it a C^2 function on all of $\mathbb{R}^d$. Let L denote the differential generator of the process $(X(t), \theta(t))$. Then

$$\begin{aligned} Lw_1(x,i) = \left(\frac{k_0}{2} + \delta_0\right)(x'D_i x)^{k_0/2+\delta_0-1}\Bigg[& 2(D_i x)' A_i x + \operatorname{tr}\big(a(x,i) D_i\big) \\ & - \frac{(2 - 2\delta_0 - k_0)(D_i x)' a(x,i) D_i x}{x' D_i x} \\ & + \frac{2(x'D_i x)^{1-k_0/2-\delta_0}}{k_0 + 2\delta_0} \sum_j \lambda_{ij}\big(x' D_j x\big)^{k_0/2+\delta_0}\Bigg]. \end{aligned} \tag{3.25}$$

As in the proof of the previous lemma, use (A2)$'$ in (3.25) and choose δ_0 sufficiently small to yield

$$Lw_1(x,i) \leq -\beta_0 w_1(x,i) + K_0 \tag{3.26}$$

for some constants $\beta_0 > 0$, $K_0 > 0$. Let N_1 be a positive integer such that $|x| \leq N_1/2$. Let N be a positive integer such that $N \geq N_1$ and $\eta'_N = \inf\{t > 0: |X^{x,i}(t)| = N\}$. Let $t'_N = t \wedge \eta'_N$. By Itô's formula

$$Ew_1\big(X^{x,i}(t'_N), \theta^i(t'_N)\big) + E\left[\int_0^{t'_N} \left\{-Lw_1\big(X^{x,i}(s), \theta^i(s)\big)\right\} ds\right] = w_1(x,i). \tag{3.27}$$

From (3.26), $K_0 - Lw_1(x,i) \geq \beta_0 w_1(x,i)$. Therefore, from (3.27) we get

$$Ew_1\big(X^{x,i}(t'_N), \theta^i(t'_N)\big) + E\left[\int_0^{t'_N} \left\{K_0 - Lw_1\big(X^{x,i}(s), \theta^i(s)\big)\right\} ds\right]$$
$$= w_1(x,i) + K_0 Et'_N.$$

Hence

$$\sup_{n \geq N_1} Ew_1\big(X^{x,i}(t'_N), \theta^i(t'_N)\big) \leq w_1(x,i) + K_0 t. \tag{3.28}$$

Next, define another Liapunov function

$$w_2(x,i) = (x' D_i x)^{k_0/2 + \delta_1}$$

with $0 < \delta_1 < \delta_0$. Then by (3.28), $w_2(X^{x,i}(t_N), \theta^i(t_N))$ is uniformly integrable for $N \geq N_1$. As before we can show that for all $x \in \mathbb{R}^d$, $i \in \Theta$

$$Lw_2(x,i) \leq -\beta_1 w_2(x,i) + K_1 \tag{3.29}$$

for some constants $\beta_1 > 0$, $K_1 > 0$. By Itô's formula

$$Ew_2\big(X^{x,i}(t'_N), \theta^i(t'_N)\big) = w_2(x,i) + E\left[\int_0^{t_N} Lw_2\big(X^{x,i}(s), \theta^i(s)\big)\, ds\right].$$

Letting $N \to \infty$, we get

$$Ew_2\big(X^{x,i}(t), \theta^i(t)\big) = w_2(x,i) + E\left[\int_0^{t} Lw_2\big(X^{x,i}(s), \theta^i(s)\big)\, ds\right]. \tag{3.30}$$

Set

$$\varphi(t) = Ew_2\big(X^{x,i}(t), \theta^i(t)\big).$$

Then using (3.29) and (3.30), we obtain

$$\frac{d}{dt}\varphi(t) \leq -\beta_1 \varphi(t) + K_1. \tag{3.31}$$

Finally, the desired result follows from (3.31) by an application of Gronwall's inequality. ∎

COROLLARY 3.2. *Under* (A1), (A2)′ *the family of transition probabilities* $\{p(t,x,i,dy \times \{j\}) \colon t \geq 0\}$ *for fixed* $(x,i) \in \mathbb{R}^d \times \Theta$ *is tight*.

Proof. This follows from (3.24) and Chebyshev's inequality. ∎

We now show that $\{p(t,x,i,dy\times\{j\})\colon t\ge 0\}$ is Cauchy in the bounded Lipschitzian metric (BL) defined on the space $\mathscr{P}(\mathbb{R}^d\times\Theta)$ of all probability measures on $\mathbb{R}^d\times\Theta$ as follows: For $P_1,P_2\in\mathscr{P}(\mathbb{R}^d\times\Theta)$

$$d_{BL}(P_1,P_2) = \sup_{f\in BL}\left|\sum_{i=1}^{N}\int_{\mathbb{R}^d} f(x,i)P_1(dx,\{i\}) - \sum_{i=1}^{N}\int_{\mathbb{R}^d} f(x,i)P_2(dx,\{i\})\right| \tag{3.32}$$

and

$$BL=\big\{f\colon\mathbb{R}^d\to\mathbb{R}\colon |f(x,i)-f(y,j)|\le|x-y|+d(i,j) \text{ and } |f(\cdot,\cdot)|\le 1\big\} \quad \text{for any } (x,i),(y,j)\in\mathbb{R}^d\times\Theta, \tag{3.33}$$

where

$$d(i,j)=\begin{cases}1, & \text{if } i\ne j\\ 0, & \text{if } i=j.\end{cases}$$

LEMMA 3.3. *Under* (A1)–(A2), $\{p(t,x,i,dy\times\{j\})\colon t\ge 0\}$ *for fixed* $(x,i)\in\mathbb{R}^d\times\Theta$ *is Cauchy in the BL metric*.

Proof. Let $f\in BL$, $x,y\in\mathbb{R}^d$, and $i,j\in\Theta$. Then for $t,s\ge 0$, we have

$$\begin{aligned}
&\big|Ef\big(Y^{x,i}(t+s)\big)-Ef\big(Y^{y,j}(t)\big)\big|\\
&\quad=\Big|E\Big[E\Big(f\big(Y^{x,i}(t+s)\big)\,|\,\mathscr{F}_s\Big)\Big]-Ef\big(Y^{y,j}(t)\big)\Big|\\
&\quad=\left|\sum_{l=1}^{N}\int_{\mathbb{R}^d}Ef\big(Y^{z,l}(t)\big)p(s,x,i,dz\times\{l\})-Ef\big(Y^{y,j}(t)\big)\right|\\
&\quad\le\sum_{l=1}^{N}\int_{\mathbb{R}^d}\big|Ef\big(Y^{z,l}(t)\big)-Ef\big(Y^{y,j}(t)\big)\big|p(s,x,i,dz\times\{l\}),
\end{aligned} \tag{3.34}$$

where $\mathscr{F}_s$ is the right continuous version of the filtration in (3.17). Let $\theta_1(\cdot)$ and $\theta_2(\cdot)$ be two independent replicas of the Markov chain $\theta(\cdot)$. Define

$$T=\inf\big\{t>0\colon \theta_1(t)=\theta_2(t)\big\}.$$

Then T is a stopping time with respect to the filtration $\{\mathscr{F}_t\}$. For a positive integer n, let $T_n=T\wedge n$. Let $\tilde\theta(\cdot)$ be defined as

$$\tilde\theta(u)=\begin{cases}\theta_1(u), & \text{for } u\ge T\\ \theta_2(u), & \text{for } u<T.\end{cases} \tag{3.35}$$

Then $\tilde\theta(\cdot)$ is also a Markov chain with the same transition law as that of $\theta_1(\cdot)$ and $\theta_2(\cdot)$. Since $Ef(Y^{w,k}(t))$ depends only on the law of the process

$Y^{w,k}(\cdot)$ and not on a particular realization, we have

$$\begin{aligned}
&\big|Ef(Y^{z,l}(t)) - Ef(Y^{y,j}(t))\big| \\
&\quad = \Big|Ef\big(X^{z,l}(t), \theta_1^l(t)\big) - Ef\big(X^{y,j}(t), \tilde{\theta}^j(t)\big)\Big| \\
&\quad = \Big|E\Big(\Big[f\big(X^{z,l}(t), \theta_1^l(t)\big) - f\big(X^{y,j}(t), \tilde{\theta}^j(t)\big)\Big] I_{\{T \le n\}}\Big) \\
&\qquad + E\Big(\Big[f\big(X^{z,l}(t), \theta_1^l(t)\big) - f\big(X^{y,j}(t), \tilde{\theta}^j(t)\big)\Big] I_{\{T > n\}}\Big)\Big| \\
&\quad \le \Big|E\Big[E\Big(\Big[f\big(X^{z,l}(t), \theta_1^l(t)\big) - f\big(X^{y,j}(t), \tilde{\theta}^j(t)\big)\Big] I_{\{T \le n\}} \mid \mathscr{F}_{T_n}\Big)\Big]\Big| \\
&\qquad + 2P(T > n). \qquad (3.36)
\end{aligned}$$

For $n < t$,

$$\begin{aligned}
&\Big|E\Big[f\big(X^{z,l}(t), \theta_1^l(t)\big) - f\big(X^{y,j}(t), \tilde{\theta}^j(t)\big) I_{\{T \le n\}} \mid \mathscr{F}_{T_n}\Big]\Big| \\
&\quad \le \Big|E\Big[f\big(X^{v,q}(t - T_n), \theta_1^l(t - T_n)\big) - f\big(X^{w,q}(t - T_n), \theta_1^l(t - T_n)\big)\Big]\Big| \\
&\quad \le E\big(\big|X^{v,q}(t - T_n) - X^{w,q}(t - T_n)\big| \wedge 2\big) \qquad (3.37)
\end{aligned}$$

conditionally on $v = X^{z,l}(T_n)$, $w = X^{y,j}(T_n)$, and $q = \theta_1^l(T_n)$.

We now claim that there exists a $r' > 0$ such that for any fixed $v, w \in \mathbb{R}^d$, $q \in \Theta$

$$E\Big[e^{r'(t - T_n)}\big|X^{v,q}(t - T_n) - X^{w,q}(t - T_n)\big|^k\Big] \le C|v - w|^k \qquad (3.38)$$

for some constant $C > 0$, where k is as defined in (A2). Taking (3.38) for granted, we get

$$\begin{aligned}
&E\big|X^{v,q}(t - T_n) - X^{w,q}(t - T_n)\big|^k \\
&\qquad \le Ce^{-r'|t-n|}|v - w|^k \to 0 \qquad \text{as } t \to \infty \text{ for any fixed } n. \qquad (3.39)
\end{aligned}$$

This implies

$$\begin{aligned}
&\big|X^{v,q}(t - T_n) - X^{w,q}(t - T_n)\big| \to 0, \\
&\text{in probability, exponentially fast, as } t \to \infty, \qquad (3.40)
\end{aligned}$$

for any fixed n. Thus from (3.36)–(3.40), we get

$$\begin{aligned}
&\big|Ef(Y^{z,l}(t)) - Ef(Y^{y,j}(t))\big| \\
&\qquad \le E\big(\big|X^{v,q}(t - T_n) - X^{w,q}(t - T_n)\big| \wedge 2\big) + 2P(T > n). \qquad (3.41)
\end{aligned}$$

Letting $t \to \infty$ first and using Lebesgue dominated convergence theorem and then $n \to \infty$ and observing that T is finite almost surely, in (3.41), we get

$$\big|Ef(Y^{z,l}(t)) - Ef(Y^{y,j}(t))\big| \to 0 \qquad \text{as } t \to \infty. \qquad (3.42)$$

Finally, using (3.42), the tightness of $\{p(t, x, i, dy \times \{j\}) \colon t \ge 0\}$, and the Lebesgue dominated convergence theorem the desired result follows from

(3.34). Thus to complete the proof of the lemma it suffices to show that there exists a $r' > 0$ such that for any bounded stopping time S and for fixed $v, w \in \mathbb{R}^d$, $q \in \Theta$,

$$E\left[e^{r'S}\left|X^{v,q}(S) - X^{w,q}(S)\right|^k\right] \le |v-w|^k, \tag{3.43}$$

where k is as defined in (A2). We can choose a suitable $r' > 0$ as in (3.13) and mimic the arguments used in the proof of the Lemma 3.1 to prove (3.43). We omit the details. ∎

THEOREM 3.1. *Under* (A1)–(A2), *the process* $Y(t) = (X(t), \theta(t))$ *is stable in distribution.*

Proof. Let $(x, i) \in \mathbb{R}^d \times \Theta$. Since the convergence in $\mathscr{P}(\mathbb{R}^d \times \Theta)$ with respect to the d_{BL} metric is equivalent to the weak convergence [3], it follows by the Cauchy property of $\{p(t, x, i, dy \times \{j\}) \colon t \ge 0\}$ that

$$p(t, x, i, dy \times \{j\}) \to \pi(dy \times \{j\})$$

weakly, as $t \to \infty$ for some $\pi \in \mathscr{P}(\mathbb{R}^d \times \Theta)$. Now for arbitrary $(z, k) \in \mathbb{R}^d \times \Theta$,

$$\begin{aligned} d_{BL}(p(t, z, k, dy \times \{j\}), \pi(dy \times \{j\})) &\\ \le d_{BL}(p(t, z, k, dy \times \{j\}), p(t, x, i, dy \times \{j\})) &\\ + d_{BL}(p(t, x, i, dy \times \{j\}), \pi(dy \times \{j\})). & \end{aligned} \tag{3.44}$$

The first term on the right side of (3.44) can be shown to tend to zero as $t \to \infty$ using the same arguments employed to prove (3.42). The second term goes to zero as $t \to \infty$. Therefore the limiting distribution is independent of the initial position. It is now easily seen, using the Chapman–Kolmogorov equation for the transition probability and the weak convergence, that π is the unique invariant measure of $Y(t)$. ∎

COROLLARY 3.3. *Suppose the system* (2.1) *is MSS. Assume* (A1) *and suppose that* $dk_0^2 < 1/\Lambda_P$ (*see Corollary* 3.1), *then the process* $Y(t) = (X(t), \theta(t))$ *is stable in distribution.*

Proof. We have seen in the proof of Corollary 3.1 that the assumed conditions imply (A2). Hence the result. ∎

Remark 3.2. If the system (2.1) is MSS and $\sigma(x, i)$ is independent of (x, i) then the conditions of the above corollary are obviously satisfied. Hence the process $Y(t)$ would be stable in distribution.

In light of Theorem 3.1 the following examples will show why the noise added system remains stable when (i) the system (2.1) is MSS; (ii) the system (2.1) is not MSS.

EXAMPLE 3.1. Let $k = 1$ in assumption (A2), and $\Theta = \{1, \ldots, N\}$, $b(x, i) = A_i x$, $\sigma_1(x, i) = x$ and $\sigma_j(x, i) = 0$ for all $j \ne 1$. Here the A_i's are constant $d \times d$ matrices where all eigenvalues of A_i's have negative real parts and σ_j is the jth column of σ. It is easy to see that (A1) is satisfied. Now notice that there exist positive definite matrices of B_i's such that $A_i' B_i + B_i A_i = -I$. Let δ^B and δ_B be the largest and the smallest, respectively, of the eigenvalues of all $\{B_i\}_{i=1}^N$. For each i, let $|\lambda_{ii}| < (1/(4(\delta^B - \delta_B)))$. Notice that since B_i depends on A_i, this condition on the λ_{ii}'s really shows how they depend on the (eigenvalues of) A_i's. Now for $x \in \mathbb{R}^d$, $i \in \Theta$,

$$
\begin{aligned}
& 2(A_i x)'(B_i x) + \sum_{j=1}^{N} \lambda_{ij}\big(x' B_j x\big) \\
& \qquad \le x'(A_i' B_i + B_i A_i)x + \delta^B |x|^2 \left(\sum_{j=1_{j \neq i}}^{N} \lambda_{ij} \right) + \delta_B |x|^2 \lambda_{ii} \\
& \qquad \le -|x|^2 + |x|^2(-\lambda_{ii})\big(\delta^B - \delta_B\big) \\
& \qquad \le -\left(1 - \frac{1}{4}\right)|x|^2
\end{aligned}
$$

This shows $E_{x,i} w(X(t), \theta(t)) \to 0$ as $t \to \infty$ where $w(x, l) = (x' B_l x)$. Since the B_i's are positive definite matrices this gives that the system defined in (2.1) is MSS.

We shall now show that the system defined in (2.6) satisfies (A2): for $x \neq y \in \mathbb{R}^d$, $i \in \Theta$, $k = 1$

$$
\begin{aligned}
& 2\big(A_i(x-y)\big)'\big(B_i(x-y)\big) - \frac{(B_i(x-y))' a(x,y,i)\big(B_i(x-y)\big)}{(x-y)' B_i (x-y)} \\
& \qquad + \operatorname{tr}\big(a(x,y,i) B_i\big) \\
& \qquad + 2\big((x-y)' B_i (x-y)\big)^{1/2} \sum_{j=1}^{N} \lambda_{ij} \big((x-y)' B_j (x-y)\big)^{1/2} \\
& \quad \le (x-y)'(A_i' B_i + B_i A_i)(x-y) - \frac{\big((x-y)' B_i (x-y)\big)^2}{(x-y)' B_i (x-y)} \\
& \qquad + \operatorname{tr}\big((x-y)(x-y)' B_i\big) + 2\delta^B |x-y|^2 \left(\sum_{j=1_{j \neq i}}^{N} \lambda_{ij} \right) \\
& \qquad + 2\delta_B |x-y|^2 \lambda_{ii} \\
& \quad \le -|x-y|^2 - (x-y)' B_i (x-y) + (x-y)' B_i (x-y) \\
& \qquad + 2|x-y|^2(-\lambda_{ii})\big(\delta^B - \delta_B\big) \\
& \quad \le -\left(1 - \frac{1}{2}\right)|x-y|^2.
\end{aligned}
$$

Thus if we take $\gamma = 1/2$ in the assumption (A2) then by Theorem 3.1 the system in (2.6) is stable in distribution.

EXAMPLE 3.2. Let $\Theta = \{1, 2\}$, $b(x, i) = A_i x$, $\sigma_1(x, i) = bx$, for some positive constant b, and $\sigma_j(x, i) = 0$ for all $j \neq 1$ and for all $i = 1, 2$. Here the A_i's are constant 2×2 matrices and σ_j is the jth column of σ. Let us take for simplicity, $A_i = d_i I_2$ for $i = 1, 2$ where the d_i's are positive constants and I_2 is the 2×2 identity matrix. Let $c = -\lambda_{11} = \lambda_{12} = \lambda_{21} = -\lambda_{22}$ for some positive constant c. It is clear that (A1) is satisfied. It is also evident that, for positive constants d_i, the system (2.1) does not satisfy the hypothesis of Proposition 2.1 and hence it is not MSS. We shall now show for some $k \in (0, 1)$ and $b^2 > 2\max(d_1, d_2)/(1-k)$ the system defined in (2.6) satisfies (A2) and hence by Theorem 3.1 the system is stable in distribution. Let the B_i's defined in assumption (A2) be the identity matrix. For $x \neq y \in \mathbb{R}^d$, $i \in \Theta$,

$$
\begin{aligned}
2\big(A_i(x-y)\big)'(x-y) &- \frac{(2-k)(x-y)'a(x,y,i)(x-y)}{(x-y)'(x-y)} \\
&+ \operatorname{tr} a(x,y,i) \\
&+ \frac{2}{k}\big((x-y)'(x-y)\big)^{1-k/2}\sum_{j=1}^{2}\lambda_{ij}\big((x-y)'(x-y)\big)^{k/2} \\
= 2d_i|x-y|^2 &- \frac{(2-k)b^2|x-y|^4}{|x-y|^2} + b^2|x-y|^2 \\
= \big(2d_i &- (2-k)b^2 + b^2\big)|x-y|^2 \\
= -\big((1-k)&b^2 - 2d_i\big)|x-y|^2.
\end{aligned}
$$

Therefore, γ in assumption (A2) can be defined as $\gamma = (1-k)b^2 - 2\max(d_1, d_2)$.

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